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NATURAL EQUIVALENCES AND DUALITIES

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Abstract. The aim of the present survey paper is to present the basic facts and principles concerning equivalences and dualities induced by pairs of right adjoint covariant, respectively contravariant, functors.

1. Introduction

The starting point of the study of equivalences, respectively dualities, between certain full subcategories of module categories was in the 50's. In that time Morita [14] and Azumaya [1] proved some important results which generalizes some classical properties of modules over rings of matrices over fields, respectively the classical duality theorem for vector spaces. More precisely, if R and S are unital rings, they proved

i) a right R-module P is finitely generated projective generator in Mod-R if and only if the covariant functor
$$\mathrm{Hom}(P, -) : \text{Mod-}R \to \text{Mod-End}_R(P)$$
is an equivalence;

ii) any equivalence F : Mod-R $\to$ Mod-S is representable, i.e. there exists a finitely generated projective generator P in Mod-R such that $S \cong \mathrm{End}_R(P)$ and the functors F and $\mathrm{Hom}_R(P, -)$ are naturally equivalent functors;

iii) If $\mathcal{C} \subseteq$ Mod-R and $\mathcal{D} \subseteq S$-Mod are full subcategories which are closed under finite products, submodules and epimorphic images such that $S \in \mathcal{D}$ then every duality $F : \mathcal{C} \to \mathcal{D}$ is representable, i.e. there exists an injective cogenerator U for $\mathcal{C}$ such that $S \cong \mathrm{End}_R(U)$, U is an injective cogenerator for S-Mod and the functors F and $\mathrm{Hom}_R(-, P)$ are naturally equivalent.

During the time, these results were generalized in various ways by many authors. We refer here to the book [7] and the papers [16], [17], [18] for surveys concerning theories about equivalences or dualities between full subcategories of module categories. Important tools in these theories are (co)tilting modules and their generalizations. These kind of modules induce some special equivalences (dualities) between some natural torsion and torsion-free

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classes. Wisbauer observed in the introduction of [17] that many results concerning tilting modules can be generalized (using similar techniques) to locally finitely generated Grothendieck categories. Tilting equivalences in Grothendieck categories were studied by Colpi [8], Gregorio [13], Castagño-Iglesias, Gómez-Torrecillas and Wisbauer [5]. Generalization of the tilting theory for (semi-)abelian categories were obtained by Colpi and Fuller [10] and by Rump [15]. Recently, a similar program for dualities between Grothendieck categories was started by Castagño-Iglesias [4].

The aim of the present survey paper is to present the basic facts and principles concerning this kind of results.

2. Equivalences

Basics. Let $\mathcal{A}$ and $\mathcal{B}$ be Grothendieck categories and $\mathrm{H} : \mathcal{A} \rightleftarrows \mathcal{B}$ a covariant functor which has a left adjoint T. We will denote by $\phi : \mathrm{TH} \to 1_{\mathcal{A}}$, respectively $\theta : 1_{\mathcal{B}} \to \mathrm{HT}$ the natural transformations which correspond to this pair of adjoint functors. We will say that an object $X \in \mathcal{A}$ is H-*static* if ϕ_X is an isomorphism. An object $Y \in \mathcal{B}$ is H-*adstatic* if θ_Y is an isomorphism. If $\mathcal{B}$ has an H-adstatic generator B, we will say that the triple $\mathfrak{H} = (\mathrm{H}, \mathrm{T}, B)$ is a *right pointed pair of adjoint functors*, and we shall speak of an $\mathfrak{H}$-static (adstatic) object rather of an H-static (adstatic) object.

In this section $\mathfrak{H}$ *will denote a right pointed pair of adjoint covariant functors* $(\mathrm{H}, \mathrm{T}, B)$ *between the Grothendieck categories* $\mathcal{A}$ *and* $\mathcal{B}$.

We associate to every pair $\mathfrak{H}$ some important classes of objects:

- $\mathrm{Gen}(\mathfrak{H}) = \{M \in \mathcal{A} \mid \phi_M \text{ is an epimorphism}\}$, the class of $\mathfrak{H}$-*generated objects*,
- $\mathrm{Faith}(\mathfrak{H}) = \{X \in \mathcal{B} \mid \theta_X \text{ is a monomorphism}\}$, the class of $\mathfrak{H}$-*faithful objects*.

Moreover, if X is an object in a Grothendieck category, we will by $\mathrm{Cogen}(X)$ the class (closed with respect isomorphisms) of subobjects of powers of X. This is the class of all X-*cogenerated* objects.

Since $\phi_{\mathrm{T}(X)}\mathrm{T}(\theta_X) = 1_{\mathrm{T}(X)}$ for all $X \in \mathcal{A}$ and $\mathrm{H}(\phi_M)\theta_{\mathrm{H}(M)} = 1_{\mathrm{H}(M)}$ for all $M \in \mathcal{A}$ (see [19]), we have the inclusions $\mathrm{T}(\mathcal{B}) \subseteq \mathrm{Gen}(\mathfrak{H})$ and $\mathrm{H}(\mathcal{A}) \subseteq \mathrm{Faith}(\mathfrak{H})$.

Lemma 2.1. *For every object* $Y \in \mathcal{B}$ *we denote*

$$\mathrm{Ann}_{\mathfrak{H}}(Y) = \sum\{Z \le Y \mid \mathrm{T}(i) = 0, \text{ where } i : Z \to Y \text{ denotes the inclusion}\}.$$

Then:

i) $\mathrm{T}(\mathrm{Ann}_{\mathfrak{H}}(Y)) = 0$;
ii) $\mathrm{Ann}_{\mathfrak{H}}(Y) = \mathrm{Ker}(\theta_Y)$
iii) $\mathrm{Ann}_{\mathfrak{H}}(-) : \mathcal{B} \to \mathcal{B}$ *is a radical.*
iv) *If* Q *is a cogenerator for* $\mathcal{A}$ *then* $\mathrm{Cogen}(\mathrm{H}(Q)) = \mathrm{Faith}(\mathfrak{H})$.

Proof. i) follows from the fact that T commutes with direct limits.

ii) Let $i : \mathrm{Ker}(\theta_Y) \to Y$ the inclusion map. Then $\mathrm{T}(\theta_Y)\mathrm{T}(i) = 0$. But $T(\theta_Y)$ is a monomorphism, hence $\mathrm{T}(i) = 0$. Then $\mathrm{Ker}(\theta_Y) \subseteq \mathrm{Ann}_{\mathfrak{H}}(Y)$.

Conversely, if $Z \overset{i}{\hookrightarrow} Y$ is a subobject such that $\mathrm{T}(i) = 0$, then $\theta_Y i = \mathrm{HT}(i)\theta_Z = 0$, hence $Z \subseteq \mathrm{Ker}(\theta_Y)$.

iii) Let $Y \in \mathcal{B}$ and $i : \mathrm{Ann}_{\mathfrak{H}}(Y/\mathrm{Ann}_{\mathfrak{H}}(Y)) \hookrightarrow Y/\mathrm{Ann}_{\mathfrak{H}}(Y)$ be the inclusion map, and $Z \overset{j}{\hookrightarrow} Y$ such that $Z/\mathrm{Ann}_{\mathfrak{H}}(Y) = \mathrm{Ann}_{\mathfrak{H}}(Y/\mathrm{Ann}_{\mathfrak{H}}(Y))$. We consider the canonical homomorphisms $\pi : Y \to Y/\mathrm{Ann}_{\mathfrak{H}}(Y)$ and $\rho : Z \to Z/\mathrm{Ann}_{\mathfrak{H}}(Y)$. Then $\mathrm{T}(\pi)\mathrm{T}(j) = \mathrm{T}(\pi j) = \mathrm{T}(i\rho) = \mathrm{T}(i)\mathrm{T}(\rho) = 0$. Moreover, since T is right exact, we have an exact sequence

$$0 = \mathrm{T}(\mathrm{Ann}_{\mathfrak{H}}(Y)) \to \mathrm{T}(Y) \overset{\mathrm{T}(\pi)}{\to} \mathrm{T}(Y/\mathrm{Ann}_{\mathfrak{H}}(Y)) \to 0.$$

Therefore, $\mathrm{T}(\pi)$ is an isomorphism. It follows that $\mathrm{T}(j) = 0$, hence $\mathrm{Ann}_{\mathfrak{H}}(Y/\mathrm{Ann}_{\mathfrak{H}}(Y)) = 0$.

iv) Let $M \in \mathcal{A}$. Then there exists a monomorphism $0 \to M \to Q^I$. Since H is left exact and preserves direct products, we have an exact sequence $0 \to \mathrm{H}(M) \to \mathrm{H}(Q)^I$, hence $\mathrm{H}(\mathcal{A}) \subseteq \mathrm{Cogen}(\mathrm{H}(Q))$.

Let $X \in \mathrm{Faith}(\mathfrak{H})$. Then $\theta_X : X \to \mathrm{HT}(X)$ is a monomorphism. Since $\mathrm{H}(\mathrm{T}(X))$ is $\mathrm{H}(Q)$-cogenerated, it follows that $X \in \mathrm{Cogen}(\mathrm{H}(Q))$, hence $\mathrm{Faith}(\mathfrak{H}) \subseteq \mathrm{Cogen}(\mathrm{H}(Q))$.

Conversely, if $X \in \mathrm{Cogen}(\mathrm{H}(Q))$ and $\iota : X \to \mathrm{H}(Q)^I$ is a monomorphism, then $\mathrm{HT}(\iota)\theta_X = \theta_{\mathrm{H}(Q)^I}\iota$ is a monomorphism, hence θ_X is a monomorphism. It follows that $\mathrm{Cogen}(\mathrm{H}(Q)) \subseteq \mathrm{Faith}(\mathfrak{H})$. □

For every $X \in \mathcal{A}$ we will denote by $\mathrm{Tr}_{\mathfrak{H}}(X)$ the subobject $\mathrm{Im}(\phi_X)$.

Lemma 2.2. *With the above notations we have:*

i) $\mathrm{Tr}_{\mathfrak{H}}(X) = \sum_{f \in \mathrm{Hom}_{\mathcal{A}}(\mathrm{T}(B),X)} \mathrm{Im}(f)$;

ii) $\mathrm{Tr}_{\mathfrak{H}}$ *is an idempotent preradical.* □

Corollary 2.3. *An object $X \in \mathcal{A}$ is $\mathfrak{H}$-generated if and only if there exists an epimorphism $\mathrm{T}(B)^{(I)} \to X$ for some set I.* □

An object $X \in \mathcal{A}$ is *$\mathfrak{H}$-presented* if there exists an exact sequence

$$0 \to U \to \mathrm{T}(B) \to X \to 0$$

with $U \in \mathrm{Gen}(\mathfrak{H})$. We will denote by $\mathrm{Pres}(\mathfrak{H})$ the class of all $\mathfrak{H}$-presented modules. Since T is right exact, we have

Lemma 2.4. $\mathrm{T}(\mathcal{B}) \subseteq \mathrm{Pres}(\mathfrak{H})$. □

w-Σ-exact functors. Let B be a generator for $\mathcal{B}$. We say that the functor H is *w-Σ-exact* if it preserves the exactness of all exact sequences $0 \to K \to T(B)^{(I)} \to L \to 0$ in $\mathcal{A}$, with $K \in \mathrm{Gen}(\mathfrak{H})$.

Theorem 2.5. *The following are equivalent for a right pointed pair $\mathfrak{H}$:*

a) $\mathrm{H} : \mathrm{Pres}(\mathfrak{H}) \rightleftarrows \mathrm{Faith}(\mathfrak{H}) : \mathrm{T}$ *is an equivalence;*

b) i) H *preserves direct sums of copies of* $\mathrm{T}(B)$,

ii) H *preserves the exactness of epimorphisms* $\beta : M \to N$*, with* $M \in \mathrm{Pres}(\mathfrak{H})$ *and* $\mathrm{Ker}(\beta) \in \mathrm{Gen}(\mathfrak{H})$*;*

c) i) H *preserves direct sums of copies of* $\mathrm{T}(B)$*,*

ii) H *is w-*Σ*-exact.*

Proof. a)⇒b) Since $B^{(I)}$ is $\mathfrak{H}$-adstatic for all sets I, we have the natural isomorphism $B^{(I)} \cong \mathrm{HT}(B^{(I)}) \cong \mathrm{H}\left(\mathrm{T}(B)^{(I)}\right)$.

Let $0 \to L \to M \to N \to 0$ be an exact sequence such that $M \in \mathrm{Pres}(\mathfrak{H})$ and $L \in \mathrm{Gen}(\mathfrak{H})$. Then we have two exact sequences $0 \to \mathrm{H}(L) \to \mathrm{H}(M) \to C \to 0$ and $0 \to X \to \mathrm{H}(N)$. The last one shows that $C \in \mathrm{Faith}(\mathfrak{H})$, so it is $\mathfrak{H}$-adstatic. Applying the functor T, respectively HT, we obtain the commutative diagrams

$$\begin{array}{ccccccccc} & & \mathrm{TH}(L) & \longrightarrow & \mathrm{TH}(M) & \longrightarrow & \mathrm{T}(C) & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \end{array},$$

respectively

$$\begin{array}{ccc} C & \longrightarrow & \mathrm{H}(N) \\ \big\downarrow & & \big\downarrow \\ \mathrm{HT}(C) & \longrightarrow & \mathrm{HTH}(N) \end{array}.$$

We observe that $\mathrm{T}(C) \to N$ is an isomorphism by snake lemma, showing that $N \in \mathrm{ImT} \subseteq \mathrm{Pres}(\mathfrak{H})$. Moreover $\mathrm{HT}(C) \to \mathrm{HTH}(N)$ is an isomorphism. Since the vertical arrows in the second diagram are isomorphisms, the top arrow in this diagram must be also an isomorphism, thus $C = \mathrm{H}(N)$.

b)⇒c) is obvious.

c)⇒a) Let $M \in \mathrm{Pres}(\mathfrak{H})$. Then there exists an exact sequence $0 \to L \to T(B)^{(I)} \to M \to 0$ with $L \in \mathrm{Gen}(\mathfrak{H})$, hence the diagram

$$\begin{array}{ccccccccc} & & \mathrm{TH}(L) & \longrightarrow & \mathrm{TH}(T(B)^{(I)}) & \longrightarrow & \mathrm{TH}(M) & \longrightarrow & 0 \\ & & \big\downarrow \psi_L & & \big\downarrow \psi_{T(B)^{(I)}} & & \big\downarrow \psi_M & & \\ 0 & \longrightarrow & L & \longrightarrow & T(B)^{(I)} & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

is commutative diagram and with exact rows. Since $T(B)^{(I)}$ is $\mathfrak{H}$-reflexive ϕ_L is an epimorphism, we deduce that M is $\mathfrak{H}$-static.

Let $X \in \mathrm{Faith}(\mathfrak{H})$. Then $\mathrm{T}(X) \in \mathrm{Pres}(\mathfrak{H})$, and it follows that it is enough to prove that X is $\mathfrak{H}$-adstatic. First we observe that $B^{(I)}$ is $\mathfrak{H}$-adstatic for all sets I. Let $\beta : B^{(I)} \to X \to 0$ be an epimorphism in $\mathcal{B}$. Then $\theta_X\beta = \mathrm{HT}(\beta)\theta_{B^{(I)}}$. By ii), $\mathrm{HT}(\beta)$ is an epimorphism, hence θ_X is an epimorphism. Since $X \in \mathrm{Faith}(\mathfrak{H})$, it follows that X is $\mathfrak{H}$-adstatic. □

$\star$-pairs. We will say that the right pointed pairs of adjoint functors $\mathfrak{H}$ is a *$\star$-pair* if it induces an equivalence

$$\mathrm{H} : \mathrm{Gen}(\mathfrak{H}) \rightleftarrows \mathrm{Faith}(\mathfrak{H}) : \mathrm{T}$$

The following lemma was proved in a more general setting in [8, Lemma 1.5] and [11, Proposition 1.1].

Lemma 2.6. *If* $\mathfrak{H}$ *is a* $\star$*-pair then:*

i) T *preserves the exactness of a monomorphism* α *in* $\mathrm{Faith}(\mathfrak{H})$ *if and only if* $\mathrm{Coker}(\alpha) \in \mathrm{Faith}(\mathfrak{H})$.
ii) H *preserves the exactness of an epimorphism* β *in* $\mathrm{Gen}(\mathfrak{H})$ *if and only if* $\mathrm{Ker}(\beta) \in \mathrm{Gen}(\mathfrak{H})$.

Proof. i) can be proved in a dual manner of the proof of a)⇒b) in Theorem 2.5.

ii) The direct implication follows by Theorem 2.5. Conversely, if $0 \to L \to M \to N \to 0$ is an exact sequence which stays exact under the application of H such that $M, N \in \mathrm{Gen}(\mathfrak{H})$ then the snake lemma for the commutative diagram

$$\begin{array}{ccccccccc} & & \mathrm{TH}(L) & \longrightarrow & \mathrm{TH}(M) & \longrightarrow & \mathrm{T}(\mathrm{H}(N)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

shows that $N \in \mathrm{Gen}(\mathfrak{H})$. □

Theorem 2.7. *The following are equivalent for a right pointed pair of adjoint functors* $\mathfrak{H} = (\mathrm{H}, \mathrm{T}, B)$.

a) $\mathfrak{H}$ *is a* $\star$*-pair;*
b) i) H *preserves direct sums of copies of* $\mathrm{T}(B)$,
 ii) $\mathrm{Gen}(\mathrm{T}(B)) = \mathrm{Pres}(\mathrm{T}(B))$,
 iii) H *respects the exactness of exact sequences in* $\mathrm{Gen}(\mathrm{T}(B))$*;*
c) i) H *preserves direct sums of copies of* $\mathrm{T}(B)$,
 ii) H *preserves the exactness of an epimorphism* $M \xrightarrow{\alpha} N \to 0$ *with* $M \in \mathrm{Gen}(\mathrm{T}(B))$ *if and only if* $\mathrm{Ker}(\alpha) \in \mathrm{Gen}(\mathrm{T}(B))$.

Proof. See [5, Theorem 2.2, Theorem 2.4] and [8, Theorem 3.2]. □

The following result was proved in [8, Theorem 3.2] and [13, Theorem 2.4].

Proposition 2.8. *If* $\mathfrak{H}$ *is a* $\star$*-pair then* $\phi : \mathrm{TH} \to 1_{\mathcal{A}}$ *is monic and* $\theta : 1_{\mathcal{B}} \to \mathrm{HT}$ *is epic. Conversely, if* ϕ_X *is monic for all* $X \in \mathrm{Gen}(\mathfrak{H})$ *and* θ_Y *is epic for all* $Y \in \mathrm{Faith}(\mathfrak{H})$ *then* $\mathfrak{H}$ *is a* $\star$*-pair.* □

An isomorphism closed subclass $\mathcal{T}$ of $\mathcal{A}$ is a *pretorsion class* in $\mathcal{A}$ if for each object $A \in \mathcal{A}$ there exist $T \in \mathcal{T}$ and a monomorphism $\mu : T \to A$ such that every homomorphism $\alpha : T' \to A$ with $T' \in \mathcal{T}$ factors through μ. An class $\mathcal{F} \subseteq \mathcal{A}$ is a *pretorsion-free class* if it satisfies the dual condition. Equivalently, $\mathcal{T}$ is a pretorsion (pretorsion-free) class if and only if A is a coreflective (reflective) full subcategory with monic counit (epic unit). These classes were used by Rump in [15] to study $\star$-objects in some additive contexts.

Proposition 2.9. *A right pointed pair $\mathfrak{H}$ is a $\star$-pair if and only if* $\mathrm{Im}(\mathrm{T})$ *is a pretorsion class in $\mathcal{A}$ and* $\mathrm{Im}(\mathrm{H})$ *is a pretorsion-free class in $\mathcal{B}$ such that the diagram of categories and functors*

$$\begin{array}{ccc} \mathcal{A} & \underset{\mathrm{H}}{\overset{\mathrm{T}}{\leftrightarrows}} & \mathcal{B} \\ \downarrow\uparrow & & \uparrow\downarrow \\ \mathrm{Im}(\mathrm{T}) & \underset{\mathrm{H}}{\overset{\mathrm{T}}{\leftrightarrows}} & \mathrm{Im}(\mathrm{H}) \end{array}$$

is commutative (the vertical arrows represent the (co)units which correspond to the (co)reflective subcategories, respectively) and the restrictions $\mathrm{H} : \mathrm{Im}(\mathrm{T}) \leftrightarrows \mathrm{Im}(\mathrm{H}) : \mathrm{T}$ *are equivalences.*

Proof. The proposition follows from the fact that the natural transformations $\theta_{\mathrm{H}} : \mathrm{H} \to \mathrm{HTH}$ and $\phi_{\mathrm{T}} : \mathrm{THT} \to \mathrm{T}$ are invertible. The reader can find details in [15, Proposition 8]. □

Quasi-tilting pairs. A $\star$-pair $\mathfrak{H}$ is called *quasi-tilting* if $\mathrm{H}(\mathrm{Coker}(\phi_M)) = 0$ for all $M \in \mathcal{A}$ and $\mathrm{T}(\mathrm{Ker}(\theta_X)) = 0$ for all $X \in \mathcal{B}$. Quasi-tilting modules were introduced by Colpi, d'Este and Tonolo in [9].

Proposition 2.10. *A $\star$-pair is a quasi-tilting pair if and only if* $(\mathrm{Im}(\mathrm{T}), \mathrm{Ker}(\mathrm{H}))$ *and* $(\mathrm{Ker}(\mathrm{T}), \mathrm{Im}(\mathrm{H}))$ *are torsion theories.*

Proof. Suppose that $\mathfrak{H}$ is a $\star$-pair. For every $X \in \mathcal{B}$ the epimorphism induced by the pretorsion-free class $\mathrm{Im}(\mathrm{H})$ is $\theta_X : X \to \mathrm{HT}(X)$. So $\mathrm{Im}(\mathrm{H})$ is a torsion-free class if and only if $\mathrm{T}(\mathrm{Ker}(\theta_X)) = 0$ for all $X \in \mathcal{B}$. A dual argument shows that $\mathrm{Im}(\mathrm{T})$ is a torsion class if and only if $\mathrm{H}(\mathrm{Coker}(\phi_M) = 0$ for all $M \in \mathcal{A}$. □

Tilting pairs. If $\mathcal{C}$ is a full subcategory of an abelian category, we will denote by $\overline{\mathcal{C}}$ the *abelian closure* of $\mathcal{C}$, i.e. the closure of $\mathcal{C}$ under finite direct sums, kernels and cokernels. For example, if $\mathfrak{H}$ is a right pointed pair, $\overline{\mathrm{Gen}(\mathfrak{H})}$ is the closure of $\mathrm{Gen}(\mathfrak{H})$ with respect subobjects, and $\overline{\mathrm{Faith}(\mathfrak{H})}$ is the closure of $\mathrm{Faith}(\mathfrak{H})$ with respect quotient objects.

Note that $\overline{\mathrm{Faith}(\mathfrak{H})}$ is a Grothendieck category, and that $\mathrm{H}(\mathcal{A}) \subseteq \mathrm{Faith}(\mathfrak{H})$. Then we can assume w.l.o.g. that $\mathcal{B} = \overline{\mathrm{Faith}(\mathfrak{H})}$.

Let $\mathfrak{H}$ be a right pointed pair of adjoint functors. We say that $\star$-pair $\mathfrak{H}$ is a *tilting pair* if $\mathcal{A} = \overline{\mathrm{Gen}(\mathfrak{H})}$.

Theorem 2.11. *A right pointed pair of adjoint functors $\mathfrak{H}$ is a $\star$-pair if and only if* $(\mathrm{H}_{|\overline{\mathrm{Gen}}(\mathrm{T}(B))}, \mathrm{T}, B)$ *is a tilting pair.* □

Since $\mathcal{A}$ is Grothendieck, it has an injective cogenerator, hence there exist the right derived functors $\mathrm{H}^{(n)}$ for H. We will denote by $\mathfrak{H}^{\perp} = \mathrm{Ker}(\mathrm{H}')$.

We have the following characterization for tilting pairs.

Theorem 2.12. *The following are equivalent for a right pointed pair $\mathfrak{H}$.*

i) *$\mathfrak{H}$ is a tilting pair;*

ii) (1) $\mathrm{T}(B)$ *is* H-*small,*
(2) $\mathfrak{H}^{\perp} = \overline{\mathrm{Gen}(\mathfrak{H})}$,
(3) $\overline{\mathrm{Gen}(\mathfrak{H})} = \mathcal{A}$.

Proof. i)⇒ii) It is enough to prove (2).

Let $M \in \mathfrak{H}^{\perp}$. By $\overline{\mathrm{Gen}(\mathfrak{H})} = \mathcal{A}$, there exists an exact sequence $0 \to M \to N \to P \to 0$ such that $N, P \in \mathrm{Gen}(\mathfrak{H})$. This sequence stays exact under F since $F'(M) = 0$. By Lemma 2.6, $M \in \mathrm{Gen}(\mathfrak{H})$.

Let M be an object in $\mathrm{Gen}(\mathfrak{H})$. Since $\mathcal{A}$ has enough injectives, there exists an exact sequence $0 \to M \to Q \to Q/M \to 0$ such that Q is injective. Then $\mathrm{H}'(Q) = 0$, hence we have the exact sequence

$$0 \to \mathrm{H}(M) \to \mathrm{H}(Q) \to \mathrm{H}(Q/M) \to \mathrm{H}'(M) \to 0.$$

But $Q \in \mathfrak{H}^{\perp}$, hence $Q \in \mathrm{Gen}(\mathfrak{H})$. By Theorem 2.7 the sequence

$$0 \to \mathrm{H}(M) \to \mathrm{H}(Q) \to \mathrm{H}(Q/M) \to 0$$

is exact, hence $M \in \mathfrak{H}^{\perp}$.

ii)⇒i) Let

$$(\sharp)\ 0 \to M \to N \to P \to 0$$

be an exact sequence in $\mathcal{A}$. If $M \in \mathrm{Gen}(\mathfrak{H})$, using (2) we obtain that $(\sharp)$ stays exact under H. Conversely, if H preserves the exactness of $(\sharp)$ and $N \in \mathrm{Gen}(\mathfrak{H})$ then $M \in \mathfrak{H}^{\perp} = \mathrm{Gen}(\mathfrak{H})$. By Theorem 2.7, $\mathfrak{H}$ is a $\star$-pair. □

Fortunately, if $\mathfrak{H}$ is a tilting pair, we can construct a "left derived functor" for T.

Proposition 2.13. [13] *Let $\mathfrak{H}$ be a tilting pair. Then there exists a functor* $\mathrm{T}' : \mathcal{B} \to \mathcal{A}$ *such that for every exact sequence* $0 \to X \to Y \to Z \to 0$ *in* $\mathcal{B}$ *we have an exact sequence*

$$0 \to \mathrm{T}'(X) \to \mathrm{T}'(Y) \to \mathrm{T}'(Z) \xrightarrow{\delta} \mathrm{T}(X) \to \mathrm{T}(Y) \to \mathrm{T}(Z) \to 0.$$

Moreover, T' *is a right adjoint for the first derived functor* H' *of* H. □

Remark 2.14. The functor T' is constructed in the following way: For every $X \in \mathcal{B}$, we consider a presentation $0 \to Z \xrightarrow{\alpha} Y \to X \to 0$ with $Z, Y \in \mathrm{Faith}(\mathfrak{H})$. Then $\mathrm{T}'(X) = \mathrm{Ker}(\mathrm{T}(\alpha))$. Therefore, since T preserves the exactness of exact sequences in $\mathrm{Faith}(\mathfrak{H})$, $\mathrm{T}'(\mathrm{Faith}(\mathfrak{H})) = 0$.

If $\mathfrak{H}$ is a tilting pair, we will consider the right pointed pair $\mathfrak{H}' = (\mathrm{H}', \mathrm{T}', B)$. Now we can enunciate "the tilting theorem".

Theorem 2.15. *Let $\mathfrak{H}$ be a tilting pair. Then*

(1) $\mathfrak{H}'$ *is a quasi-tilting pair,*
(2) $(\mathrm{Im}(\mathrm{T}), \mathrm{Ker}(\mathrm{H})) = (\mathrm{Ker}(\mathrm{H}'), \mathrm{Im}(\mathrm{T}'))$ *is a torsion theory in* $\mathcal{A}$,
(3) $(\mathrm{Ker}(\mathrm{T}), \mathrm{Im}(\mathrm{H})) = (\mathrm{Im}(\mathrm{H}'), \mathrm{Ker}(\mathrm{T}'))$ *is a torsion theory in* $\mathcal{B}$,
(4) $\mathrm{H} : \mathrm{Im}(\mathrm{T}) \leftrightarrows \mathrm{Im}(\mathrm{H}) : \mathrm{T}$ *and* $\mathrm{H}' : \mathrm{Ker}(\mathrm{H}) \leftrightarrows \mathrm{Ker}(\mathrm{T}) : \mathrm{T}'$ *are equivalences.*

Proof. The reader can find complete proofs in [13, Section 3] and [15] (see also [12]). We present here (as a sample) the proof for the equality $\mathrm{Ker}(\mathrm{T}) = \mathrm{Im}(\mathrm{H}')$.

Let $X \in \mathrm{Ker}(\mathrm{T})$. There exists an exact sequence $0 \to Z \overset{\alpha}{\to} Y \to X \to 0$ with $Z, Y \in \mathrm{Faith}(\mathfrak{H})$. Then we have an exact sequence

$$0 \to \mathrm{HT}'(X) \to \mathrm{HT}(Z) \to \mathrm{HT}(Y) \to \mathrm{H}'(\mathrm{T}'(X)) \to \mathrm{H}'(\mathrm{T}(Z)) = 0.$$

Since $\mathrm{H} : \mathrm{Gen}(\mathfrak{H}) \leftrightarrows \mathrm{Faith}(\mathfrak{H}) : \mathrm{T}$ is an equivalence, $X = \mathrm{Coker}(\alpha) \cong \mathrm{Coker}(\mathrm{HT}(\alpha)) = \mathrm{H}'(\mathrm{T}'(X))$ (moreover, this isomorphism is natural). Then $\mathrm{Ker}(\mathrm{T}) \subseteq \mathrm{Im}(\mathrm{H}')$.

If $X = \mathrm{H}'(L)$, we consider an exact sequence $0 \to L \to Q \to Q/L \to 0$ with Q an injective object in $\mathcal{A}$. Then the sequence

$$\mathrm{TH}(Q) \to \mathrm{TH}(Q/L) \to \mathrm{TH}'(L) \to 0$$

is exact. $\mathrm{H} : \mathrm{Gen}(\mathfrak{H}) \leftrightarrows \mathrm{Faith}(\mathfrak{H}) : \mathrm{T}$ is an equivalence and $Q, Q/L \in \mathrm{Gen}(FH)$, it follows that $\mathrm{T}(\mathrm{H}'(L)) = 0$. ☐

Representable tilting($\star$)-pairs. Let $\mathfrak{H} = (\mathrm{H}, \mathrm{T}, B)$ be a tilting pair, and $P = \mathrm{T}(B)$. If S is the endomorphism ring of P and T is the endomorphism ring of B then there is a ring isomorphism $S \cong T$ and an equivalence S-Mod $\approx$ T-Mod which are induced by $\mathfrak{H}$. Moreover, P induces a right pointed pair $\mathfrak{H}_P = (\mathrm{H}_P, \mathrm{T}_P, S)$, where $\mathrm{H}_P = \mathrm{Hom}(P, -) : \mathcal{A} \to S\text{-Mod} : \mathrm{T}_P$ are the canonical adjoint functors. A similar pointed pair $\mathfrak{H}_B$ is induced by B. Therefore, we have commutative diagrams

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathrm{H}} & \mathcal{B} \\ {\scriptstyle \mathrm{H}_P}\downarrow & & \downarrow{\scriptstyle \mathrm{H}_B} \\ S\text{-Mod} & \xrightarrow{\approx} & T\text{-Mod} \end{array} \qquad \begin{array}{ccc} \mathcal{A} & \xleftarrow{\mathrm{T}} & \mathcal{B} \\ {\scriptstyle \mathrm{T}_P}\uparrow & & \uparrow{\scriptstyle \mathrm{T}_B} \\ S\text{-Mod} & \xleftarrow{\approx} & T\text{-Mod} \end{array}$$

Theorem 2.16. *Let $\mathfrak{H}$ be a $\star$-pair such that P is w-Σ-quasi-projective and self-small. Then in the diagram*

$$\begin{array}{ccc} \mathrm{Pres}(P) & \underset{\mathrm{H}}{\overset{\mathrm{T}}{\leftrightarrows}} & \mathrm{Faith}(\mathfrak{H}) \\ {\scriptstyle \mathrm{H}_P}\downarrow\uparrow{\scriptstyle \mathrm{T}_P} & & {\scriptstyle \mathrm{H}_B}\uparrow\downarrow{\scriptstyle \mathrm{T}_B} \\ \mathrm{Faith}(\mathfrak{H}_P) & \underset{\approx}{\overset{\approx}{\leftrightarrows}} & \mathrm{Faith}(\mathfrak{H}_P) \end{array}$$

all (functor) arrows are equivalences and this diagram in commutative.

3. Dualities

Right pointed pairs of contravariant functors. Let $\mathcal{A}$ and $\mathcal{B}$ be Grothendieck categories and $\mathrm{F} : \mathcal{A} \rightleftarrows \mathcal{B} : \mathrm{G}$ a pair of contravariant functors which are adjoint on the right, i.e. there are natural isomorphisms

$$\eta_{X,Y} : \mathrm{Hom}_{\mathcal{A}}(X, \mathrm{G}(Y)) \to \mathrm{Hom}_{\mathcal{B}}(Y, \mathrm{F}(X)) \text{ for all } X \in \mathcal{A} \text{ and } Y \in \mathcal{B}.$$

Then they induce two natural transformations

$$\delta : 1_{\mathcal{A}} \to \mathrm{GF} \text{ and } \zeta : 1_{\mathcal{B}} \to \mathrm{FG}$$

defined by

$$\delta_X = \eta^{-1}_{X,\mathrm{F}(X)}(1_{\mathrm{F}(X)}) \text{ and } \zeta_Y = \eta^{-1}_{\mathrm{G}(Y),Y}(1_{\mathrm{G}(Y)}).$$

Moreover, we have the identities

$$\mathrm{F}(\delta_X) \circ \zeta_{\mathrm{F}(X)} = 1_{\mathrm{F}(X)} \text{ and } \mathrm{G}(\zeta_Y) \circ \delta_{\mathrm{G}(Y)} = 1_{\mathrm{G}(Y)} \text{ for all } X \in \mathcal{A} \text{ and } Y \in \mathcal{B}.$$

An object X is called δ (respectively, ζ)-*reflexive* if δ_X (respectively, ζ_X) is an isomorphism. We will denote by Refl_δ (Refl_ζ) the classes of all objects $X \in \mathcal{A}$ ($X \in \mathcal{B}$) such that X is an δ-reflexive object (X is an ζ-reflexive object).

We also fix a ζ-reflexive generator V for $\mathcal{B}$, and the triple $\mathfrak{D} = (\mathrm{F}, \mathrm{G}, V)$ will be called a *right pointed pair of contravariant functor*. Let $Q = \mathrm{G}(V)$. Then $\mathrm{add}(Q) \subseteq \mathrm{Refl}_\delta$ and $\mathrm{add}(V) \subseteq \mathrm{Refl}_\zeta$. We will denote by Faith_δ (respectively, Faith_ζ) the classes of all objects $X \in \mathcal{A}$ (respectively, $X \in \mathcal{B}$) such that δ_X (respectively, ζ_X) is a monomorphism. We will call the objects in the classes Faith_δ and Faith_ζ as $\mathfrak{D}$-faithful objects.

In this section $\mathfrak{D}$ *will always denote a right pointed pair* $(\mathrm{F}, \mathrm{G}, V)$ *of contravariant functors.*

Lemma 3.1. *Let* $\mathrm{F} : \mathcal{A} \rightleftarrows \mathcal{B} : \mathrm{G}$ *a pair of contravariant functors which are adjoint on the right. Then*

(a) $\mathrm{F}(\mathcal{A}) \subseteq \mathrm{Faith}_\zeta$ *and* $\mathrm{G}(\mathcal{B}) \subseteq \mathrm{Faith}_\delta$.
(b) *The classes* Faith_δ *and* Faith_ζ *are closed with respect subobjects.*
(c) *Let* $X \in \mathrm{Refl}_\delta$ *and* $i : Y \to X$ *an subobject such that* $\mathrm{F}(i)$ *is an epimorphism. Then* $Y \in \mathrm{Refl}_\delta$ *if and only if* $X/Y \in \mathrm{Faith}_\delta$.

Proof. (a) It follows from the identities $\mathrm{F}(\delta_X) \circ \zeta_{\mathrm{F}(X)} = 1_{\mathrm{F}(X)}$, respectively $\mathrm{G}(\zeta_Y) \circ \delta_{\mathrm{G}(Y)} = 1_{\mathrm{G}(Y)}$.

(b) Let $X \in \mathrm{Faith}_\delta$ and $i : Y \to X$ an subobject of X. The following diagram is commutative

$$\begin{array}{ccc} Y & \xrightarrow{\delta_Y} & \mathrm{GF}(Y) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \mathrm{GF}(i)} \\ X & \xrightarrow{\delta_X} & \mathrm{GF}(X) \end{array}$$

so we have $\delta_X \circ i = \mathrm{GF}(i) \circ \delta_Y$. Since δ_X and i are monomorphisms we obtain that δ_Y is monomorphism, hence $Y \in \mathrm{Faith}_\delta$. This shows that Faith_δ is closed with respect subobjects.

The proof for Faith_ζ is similar.

(c) The sequence $0 \to Y \xrightarrow{i} X \xrightarrow{p} X/Y \to 0$ is exact, because $i : Y \to X$ is a monomorphism. Since $\mathrm{F}(i)$ is an epimorphism, the sequence

$$0 \to \mathrm{F}(X/Y) \xrightarrow{\mathrm{F}(p)} \mathrm{F}(X) \xrightarrow{\mathrm{F}(i)} \mathrm{F}(Y) \to 0$$

is also exact, hence

$$0 \to \mathrm{GF}(Y) \stackrel{\mathrm{GF}(i)}{\to} \mathrm{GF}(X) \stackrel{\mathrm{GF}(p)}{\to} \mathrm{GF}(X/Y)$$

is an exact sequence. The conclusion follows applying Snake-Lemma to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \stackrel{i}{\longrightarrow} & X & \stackrel{p}{\longrightarrow} & X/Y & \longrightarrow & 0 \\ & & \delta_Y \big\downarrow & & \big\downarrow \delta_X & & \big\downarrow \delta_{X/Y} & & , \\ 0 & \longrightarrow & \mathrm{GF}(Y) & \stackrel{\mathrm{GF}(i)}{\longrightarrow} & \mathrm{GF}(X) & \stackrel{\mathrm{GF}(p)}{\longrightarrow} & \mathrm{GF}(X/Y) & & \end{array}$$

which is commutative and with exact rows. □

w-π_f-exact functors. We will denote by $\mathrm{cop}_\delta(Q)$ the class of all objects $X \in \mathcal{A}$ such that there exists an exact sequence $0 \to X \to Q^n \to Y \to 0$ with $Y \in \mathrm{Faith}_\delta$. We will say that F is *w-π_f-exact* if it is exact with respect exact sequences $0 \to X \to Q^n \to Y \to 0$ with $Y \in \mathrm{Faith}_\delta$.

Corollary 3.2. *Let* $\mathrm{F} : \mathcal{A} \rightleftarrows \mathcal{B} : \mathrm{G}$ *be a pair of contravariant functors which are adjoint on the right,* $V \in \mathrm{Refl}_\zeta$ *and* $\mathrm{G}(V) = Q$*. Suppose that* F *is* w-π_f*-exact. Then*

(a) $\mathrm{cop}_\delta(Q) \subseteq \mathrm{Refl}_\delta$
(b) $\mathrm{F}(\mathrm{cop}_\delta(Q)) \subseteq \mathrm{gen}(V)$

Proof. (a) Let $X \in \mathrm{cop}_\delta(Q)$. There exists an exact sequence

$$0 \to X \stackrel{f}{\to} Q^n \stackrel{g}{\to} Y \to 0$$

with $Y \in \mathrm{Faith}_\delta$. Then, the sequence

$$0 \to \mathrm{F}(Y) \stackrel{\mathrm{F}(g)}{\to} \mathrm{F}(Q^n) \stackrel{\mathrm{F}(f)}{\to} \mathrm{F}(X) \to 0$$

is exact, hence

$$0 \to \mathrm{GF}(X) \stackrel{\mathrm{GF}(f)}{\to} \mathrm{GF}(Q^n) \stackrel{\mathrm{GF}(g)}{\to} \mathrm{GF}(Y)$$

is exact.

Since $\delta : 1_\mathcal{A} \to \mathrm{GF}$ is a natural transformation, we have the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \stackrel{f}{\longrightarrow} & Q^n & \stackrel{g}{\longrightarrow} & Y & \longrightarrow & 0 \\ & & \delta_X \big\downarrow & & \big\downarrow \delta_{Q^n} & & \big\downarrow \delta_Y & & \\ 0 & \longrightarrow & \mathrm{GF}(X) & \stackrel{\mathrm{GF}(f)}{\longrightarrow} & \mathrm{GF}(Q^n) & \stackrel{\mathrm{GF}(g)}{\longrightarrow} & \mathrm{GF}(Y) & & \end{array}$$

commutative, with exact sequences. From Snake Lemma we obtain that $X \in \mathrm{Refl}_\delta$.

(b) Let $X \in \mathrm{cop}_\delta(Q)$. There is an exact sequence $0 \to X \stackrel{f}{\to} Q^n \stackrel{g}{\to} Y \to 0$ with $Y \in \mathrm{Faith}_\delta$. Then the sequence $0 \to \mathrm{F}(Y) \stackrel{\mathrm{F}(g)}{\to} \mathrm{F}(Q^n) \stackrel{\mathrm{F}(f)}{\to} \mathrm{F}(X) \to 0$

is exact, hence $V^n \to \mathrm{F}(X) \to 0$ is exact (because $\mathrm{F}(Q^n) = \mathrm{F}(\mathrm{G}(V)^n) \cong (\mathrm{FG}(V))^n \cong V^n$). So, we obtain that $\mathrm{F}(X) \in \mathrm{gen}(V)$. □

Theorem 3.3. *Let* $\mathrm{F} : \mathcal{A} \rightleftarrows \mathcal{B} : \mathrm{G}$ *be a pair of contravariant functors which are adjoint on the right. If* $V \in Refl_\zeta$ *and* $\mathrm{G}(V) = Q$*, the following are equivalent:*

(a) F *is* w-π_f*-exact.*
(b) $\mathrm{F} : \mathrm{cop}_\delta(Q) \rightleftarrows \mathrm{gen}(V) \cap \mathrm{Faith}_\zeta : \mathrm{G}$ *is a duality.*

Proof. (*a*) ⇒ (*b*) From Lemma 3.1 a) and Corollary 3.2 b) we have $\mathrm{F}(\mathrm{cop}_\delta(Q)) \subseteq \mathrm{gen}(V) \cap \mathrm{Faith}_\zeta$, hence F is well-defined. From Corollary 3.2a) we obtain that $\mathrm{cop}_\delta(Q) \subseteq \mathrm{Refl}_\delta$.

Let $X \in \mathrm{gen}(V) \cap \mathrm{Faith}_\zeta$. Then there is an epimorphism $p : V^n \to X$ and the sequence $0 \to \mathrm{Ker}(p) \stackrel{i}{\to} V^n \stackrel{p}{\to} X \to 0$ is exact, hence the sequence

$$0 \to \mathrm{G}(X) \stackrel{\mathrm{G}(p)}{\to} \mathrm{G}(V^n) \stackrel{\mathrm{G}(i)}{\to} \mathrm{Im}(\mathrm{G}(i)) \to 0$$

is exact. Since $\mathrm{Im}(\mathrm{G}(i))$ is a subobject of $\mathrm{G}(\mathrm{Ker}(p))$ we obtain, by Lemma 3.1, that $\mathrm{Im}(\mathrm{G}(i)) \in \mathrm{Faith}_\delta$. It follows that $\mathrm{G}(X) \in \mathrm{cop}_\delta(Q)$ (because $\mathrm{G}(V^n) \cong Q^n$), hence G is well-defined.

From (a) we have $\mathrm{FG}(p)$ is an epimorphism and from the fact that V is ζ-reflexive it follows that ζ_X is an epimorphism. Because $X \in \mathrm{Faith}_\zeta$ we have $X \in \mathrm{Refl}_\zeta$ hence $\mathrm{gen}(V) \cap \mathrm{Faith}_\zeta \subseteq \mathrm{Refl}_\zeta$.

(*b*) ⇒ (*a*) Let $0 \to X \stackrel{f}{\to} Q^n \stackrel{g}{\to} Y \to 0$ be an exact sequence with $Y \in \mathrm{Faith}_\delta$. From (b) we obtain that $\mathrm{Im}(\mathrm{F}(f)) \in \mathrm{gen}(V) \cap \mathrm{Faith}_\zeta \subseteq \mathrm{Refl}_\zeta$, hence $\mathrm{F}(f)$ is an epimorphism by [3, Lemma 2.2(b)], and it follows that the sequence $0 \to \mathrm{F}(Y) \stackrel{\mathrm{F}(g)}{\to} \mathrm{F}(Q^n) \stackrel{\mathrm{F}(f)}{\to} \mathrm{F}(X) \to 0$ is exact. □

Costar pairs. We say that the pair $\mathfrak{D}$ is *costar* if

$$\mathrm{F} : \mathrm{F}^{-1}(\mathrm{gen}(V)) \cap \mathrm{Faith}_\delta \rightleftarrows \mathrm{gen}(V) \cap \mathrm{Faith}_\zeta : G$$

is a duality.

Theorem 3.4. *The following are equivalent for a pair* $\mathfrak{D}$*:*

(a) $\mathfrak{D}$ *is a costar pair;*
(b) (1) $\mathrm{F} : \mathrm{cop}_\delta(Q) \rightleftarrows \mathrm{gen}(V) \cap \mathrm{Faith}_\zeta : \mathrm{G}$ *is a duality and*
    (2) $\mathrm{cop}_\delta(Q) = \mathrm{F}^{-1}(\mathrm{gen}(V)) \cap \mathrm{Faith}_\delta$*;*
(c) (1) δ_X *is an epimorphism for all* $X \in \mathrm{F}^{-1}(\mathrm{gen}(V))$ *and*
    (2) ζ_X *is an epimorphism for all* $X \in \mathrm{gen}(V)$*;*
(d) F *preserves the exactness of an exact sequence*

$$0 \to X \to Q^n \to Y \to 0$$

if and only if $Y \in \mathrm{Faith}_\delta$*.*

Proof. (a)⇒(b) Let $X \in \mathrm{F}^{-1}(\mathrm{gen}(V)) \cap \mathrm{Faith}_\delta$. Then there is an exact sequence $V^n \stackrel{f}{\to} \mathrm{F}(X) \to 0$ hence there is an exact sequence

$$0 \to X \stackrel{\mathrm{G}(f)}{\to} Q^n \to Q^n/X \to 0.$$

Since $X \in \mathrm{Refl}_\delta$, $\mathrm{F}(\mathrm{G}(f))$ is epimorphism (because $\mathrm{F}(X) \in \mathrm{Refl}_\zeta$) and $Q^n \in \mathrm{Refl}_\delta$ we obtain, by Lemma 3.1 c), that $Q^n/X \in \mathrm{Faith}_\delta$. It follows that $X \in \mathrm{cop}_\delta(Q)$ hence $\mathrm{F}^{-1}(\mathrm{gen}(V)) \cap \mathrm{Faith}_\delta \subseteq \mathrm{cop}_\delta(Q)$.

Let $X \in \mathrm{cop}_\delta(Q)$. Then there is an exact sequence

$$0 \to X \xrightarrow{f} Q^n \xrightarrow{g} Y \to 0$$

with $Y \in \mathrm{Faith}_\delta$. Since $\mathrm{ImF}(f) \in \mathrm{gen}(V) \cap \mathrm{Faith}_\zeta \subseteq \mathrm{Refl}_\zeta$ we have, from [3, Lemma 2.2(b)], that $\mathrm{F}(f)$ is an epimorphism, hence

$$0 \to \mathrm{F}(Y) \xrightarrow{\mathrm{F}(g)} \mathrm{F}(Q^n) \xrightarrow{\mathrm{F}(f)} \mathrm{F}(X) \to 0$$

is exact. It follows that $V^n \to \mathrm{F}(X) \to 0$ is exact, hence $X \in \mathrm{F}^{-1}(\mathrm{gen}(V))$. But $X \in \mathrm{Faith}_\delta$ (because f is monomorphism and Q^n is δ-reflexive), so we have $X \in \mathrm{F}^{-1}(\mathrm{gen}(V)) \cap \mathrm{Faith}_\delta$, hence $\mathrm{cop}_\delta(Q) \subseteq \mathrm{F}^{-1}(\mathrm{gen}(V)) \cap \mathrm{Faith}_\delta$.

Therefore, $\mathrm{cop}_\delta(Q) = \mathrm{F}^{-1}(\mathrm{gen}(V)) \cap \mathrm{Faith}_\delta$. Using (a), we obtain that the pair $\mathrm{F} : \mathrm{cop}_\delta(Q) \rightleftarrows \mathrm{gen}(V) \cap \mathrm{Faith}_\zeta : \mathrm{G}$ is a duality.

(b)⇒(a) is obvious.

(b)⇒(c) Let $X \in \mathrm{F}^{-1}(\mathrm{gen}(V))$. From (b) we have that $\mathrm{F}(X)$ is ζ-reflexive, so $\mathrm{F}(\delta_X)$ is an isomorphism. Applying the functor F to the exact sequence

$$0 \to \mathrm{Ker}(\delta_X) \xrightarrow{i} X \xrightarrow{p} X/\mathrm{Ker}(\delta_X) \to 0$$

we obtain the exact sequence

$$0 \to \mathrm{F}(X/\mathrm{Ker}(\delta_X) \xrightarrow{\mathrm{F}(p)} \mathrm{F}(X) \xrightarrow{\mathrm{F}(i)} \mathrm{F}(\mathrm{Ker}(\delta_X)).$$

Let denote $X/\mathrm{Ker}(\delta_X)$ by $\overline{X}$. Since $\mathrm{F}(i) = 0$, it follows that $\mathrm{F}(p)$ is an isomorphism hence $\mathrm{F}(X) \cong \mathrm{F}(\overline{X})$, so $\overline{X} \in \mathrm{F}^{-1}(\mathrm{gen}(V))$. Because $\overline{X} \cong \mathrm{Im}(\delta_X)$, $\mathrm{Im}(\delta_X)$ is a subobject of $\mathrm{GF}(X)$ and $\mathrm{GF}(X) \in \mathrm{Faith}_\delta$ we obtain that $\overline{X} \in \mathrm{Faith}_\delta$ and it follows, by (b), that $\delta_{\overline{X}}$ is an isomorphism.

Therefore δ_X is an epimorphism, since we have the equality $\delta_{\overline{X}} \circ p = \mathrm{GF}(p) \circ \delta_X$, where $\delta_{\overline{X}}$ and $\mathrm{GF}(p)$ are isomorphisms (recall that $\mathrm{F}(p)$ is an isomorphism) and p is an epimorphism.

Let $X \in \mathrm{gen}(V)$. Then there is an epimorphism $g : V^n \to X$ and the following diagram is commutative:

$$\begin{array}{ccc} V^n & \xrightarrow{\zeta_{V^n}} & \mathrm{FG}(V^n) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \mathrm{FG}(g)} \\ X & \xrightarrow{\zeta_X} & \mathrm{FG}(X). \end{array}$$

Because the sequence $0 \to \mathrm{Ker}(g) \xrightarrow{i} V^n \xrightarrow{g} X \to 0$ is exact, the sequence

$$0 \to \mathrm{G}(X) \xrightarrow{\mathrm{G}(g)} Q^n \xrightarrow{\mathrm{G}(i)} \mathrm{Im}(\mathrm{G}(i)) \to 0$$

is also exact. But $\mathrm{Im}(\mathrm{G}(i))$ is a subobject of $\mathrm{G}(\mathrm{Ker}(g))$ and $\mathrm{G}(\mathrm{Ker}(g)) \in \mathrm{Faith}_\delta$ hence $\mathrm{Im}(\mathrm{G}(i)) \in \mathrm{Faith}_\delta$. Since F is *$w$-$\pi_f$-exact* (by Theorem 3.3) the sequence

$$0 \to \mathrm{F}(\mathrm{Im}(\mathrm{G}(i))) \stackrel{\mathrm{FG}(i)}{\to} \mathrm{F}(Q^n) \stackrel{\mathrm{FG}(g)}{\to} \mathrm{FG}(X) \to 0$$

is exact, hence $\mathrm{FG}(g)$ is an epimorphism.

Since $\zeta_X \circ g = \mathrm{FG}(g) \circ \zeta_{V^n}$, $\mathrm{FG}(g)$ is an epimorphism and ζ_{V^n} is an isomorphism it follows that ζ_X is an epimorphism.

(c)⇒(d) Let $0 \to X \stackrel{f}{\to} Q^n \stackrel{g}{\to} Y \to 0$ be an exact sequence. It follows that the sequence

$$0 \to \mathrm{F}(Y) \stackrel{\mathrm{F}(g)}{\to} \mathrm{F}(Q^n) \stackrel{\mathrm{F}(f)}{\to} \mathrm{F}(X) \to 0$$

is exact. Since $\mathrm{F}(X) \in \mathrm{gen}(V)$ we have that δ_X is an epimorphism. The conclusion will follow applying the Snake-Lemma to the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \stackrel{f}{\longrightarrow} & Q^n & \stackrel{g}{\longrightarrow} & Y & \longrightarrow & 0 \\ & & \delta_X \downarrow & & \downarrow \delta_{Q^n} & & \downarrow \delta_Y & & . \\ 0 & \longrightarrow & \mathrm{GF}(X) & \stackrel{\mathrm{GF}(f)}{\longrightarrow} & \mathrm{GF}(Q^n) & \stackrel{\mathrm{GF}(g)}{\longrightarrow} & \mathrm{GF}(Y) & & \end{array}$$

Conversely, let $0 \to X \stackrel{f}{\to} Q^n \stackrel{g}{\to} Y \to 0$ be an exact sequence with $Y \in \mathrm{Faith}_\delta$. Since $\mathrm{Im}(\mathrm{F}(f)) \in \mathrm{gen}(V) \cap \mathrm{Faith}_\zeta$ we obtain, by (c), that $\mathrm{Im}(\mathrm{F}(f)) \in \mathrm{Refl}_\zeta$. It follows, from [3, Lemma 2.2(b)], that $\mathrm{F}(f)$ is an epimorphism, hence the sequence $0 \to \mathrm{F}(Y) \stackrel{\mathrm{F}(g)}{\to} \mathrm{F}(Q^n) \stackrel{\mathrm{F}(f)}{\to} \mathrm{F}(X) \to 0$ is exact.

(d)⇒(b) By Theorem 3.3, the pair $\mathrm{F} : \mathrm{cop}_\delta(Q) \rightleftarrows \mathrm{gen}(V) \cap \mathrm{Faith}_\zeta : \mathrm{G}$ is a duality.

Let $X \in \mathrm{cop}_\delta(Q)$. Then there is an exact sequence $0 \to X \stackrel{f}{\to} Q^n \stackrel{g}{\to} Y \to 0$ with $Y \in \mathrm{Faith}_\delta$. Then the sequence

$$0 \to \mathrm{F}(Y) \stackrel{\mathrm{F}(g)}{\to} \mathrm{F}(Q^n) \stackrel{\mathrm{F}(f)}{\to} \mathrm{F}(X) \to 0$$

is exact and it follows that $F(X) \in \mathrm{gen}(V)$. Since f and δ_{Q^n} are monomorphisms, δ_X monomorphism hence $X \in \mathrm{F}^{-1}(\mathrm{gen}(V)) \cap \mathrm{Faith}_\delta$.

If $X \in \mathrm{F}^{-1}(\mathrm{gen}(V)) \cap \mathrm{Faith}_\delta$, then there exists an epimorphism $g : V^n \to \mathrm{F}(X)$. Set $f := \mathrm{G}(g) \circ \delta_X : X \to \mathrm{G}(V^n)$. Because δ_X and $\mathrm{G}(g)$ are monomorphisms the sequence $0 \to X \stackrel{f}{\to} Q^n \stackrel{p}{\to} \mathrm{Coker}(f) \to 0$ is exact. Since $\mathrm{F}(f) = \mathrm{F}(\delta_X) \circ \mathrm{FG}(g)$, $\mathrm{F}(\delta_X)$ is an isomorphism (because $\mathrm{F}(X) \in \mathrm{gen}(V) \cap \mathrm{Faith}_\zeta \subseteq \mathrm{Refl}_\zeta$) and $\mathrm{FG}(g)$ is an epimorphism (by $\zeta_{\mathrm{F}(X)} \circ g = \mathrm{FG}(g) \circ \zeta_{V^n}$), we have that $\mathrm{F}(f)$ is an epimorphism. Therefore,

$$0 \to \mathrm{F}(\mathrm{Coker}(f)) \stackrel{\mathrm{F}(p)}{\to} \mathrm{F}(Q^n) \stackrel{f}{\to} \mathrm{F}(X) \to 0$$

is an exact sequence. By (d), we obtain $\mathrm{Coker}(f) \in \mathrm{Faith}_\delta$ and it follows that $X \in \mathrm{cop}_\delta(Q)$. □

f-cotilting pairs. The next result describes another kind of dualities induced by right pointed pairs. If it is happens in the classical context of contravariant functors induced by a module Q, Wisbauer called this module f-cotilting, [18].

Theorem 3.5. *The following are equivalent for a pair* $\mathfrak{D}$*:*

(a) $\mathrm{F} : \mathrm{cog}(Q) \rightleftarrows \mathrm{gen}(V) \cap \mathrm{Faith}_\zeta : \mathrm{G}$ *is a duality;*
(b) i) $\mathrm{cog}(Q) = \mathrm{cop}_\delta(Q)$*;*
    ii) F *is* w-π_f*-exact* .

Proof. (a)⇒(b) Let $0 \to X \xrightarrow{f} Q^n \xrightarrow{g} Y \to 0$ be an exact sequence, with $Y \in \mathrm{Faith}_\delta$. Since $\mathrm{ImF}(f) \in \mathrm{gen}(V)$ and $\mathrm{ImF}(f)$ is a subobject of $\mathrm{F}(X)$ we have that $\mathrm{ImF}(f) \in \mathrm{Refl}_\zeta$. It follows, from [3, Lemma 2.2(b)], that $\mathrm{F}(f)$ is an epimorphism, hence the sequence $0 \to \mathrm{F}(Y) \xrightarrow{\mathrm{F}(g)} \mathrm{F}(Q^n) \xrightarrow{\mathrm{F}(f)} \mathrm{F}(X) \to 0$ is exact. Then (ii) is true.

To prove (i), we consider an object $X \in \mathrm{cog}(Q)$. Since $\mathrm{F}(X) \in \mathrm{gen}(V) \cap \mathrm{Faith}_\zeta$ and F is w-π_f-exact we have, by Theorem 3.3, $\mathrm{GF}(X) \in \mathrm{cop}_\delta(Q)$. Then there exists an exact sequence $0 \to \mathrm{GF}(X) \xrightarrow{f} Q^n \xrightarrow{g} Y \to 0$ with $Y \in \mathrm{Faith}_\delta$. Since $X \in \mathrm{cog}(Q) \subseteq \mathrm{Refl}_\delta$, the sequence $0 \to X \xrightarrow{f\circ\delta_X} Q^n \xrightarrow{g} Y \to 0$ is exact. Then $X \in \mathrm{cop}_\delta(Q)$.

The reverse inclusion is obvious.

(b)⇒(a) Let $X \in \mathrm{cog}(Q)$. Then, by (b), there is an exact sequence

$$0 \to X \xrightarrow{f} Q^n \xrightarrow{g} Y \to 0$$

such that $Y \in \mathrm{Faith}_\delta$ and the sequence

$$0 \to \mathrm{F}(Y) \xrightarrow{\mathrm{F}(g)} \mathrm{F}(Q^n) \xrightarrow{\mathrm{F}(f)} \mathrm{F}(X) \to 0$$

is exact. Then $\mathrm{F}(X) \in \mathrm{gen}(V)$. By Lemma 3.1, we have $\mathrm{F}(X) \in \mathrm{Faith}_\delta$ hence F is well-defined.

Applying Snake-Lemma to the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{f} & Q^n & \xrightarrow{g} & Y & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \delta_X} & & \downarrow{\scriptstyle \delta_{Q^n}} & & \downarrow{\scriptstyle \delta_Y} & & \\ 0 & \longrightarrow & \mathrm{GF}(X) & \xrightarrow{\mathrm{GF}(f)} & \mathrm{GF}(Q^n) & \xrightarrow{\mathrm{GF}(g)} & \mathrm{GF}(Y) & & \end{array},$$

we obtain that $X \in \mathrm{Refl}_\delta$, hence $\mathrm{cog}(Q) \subseteq \mathrm{Refl}_\delta$.

Let $X \in \mathrm{gen}(V) \cap \mathrm{Faith}_\zeta$. Then there is an epimorphism $V^n \xrightarrow{f} X \to 0$, hence $\mathrm{G}(X) \in \mathrm{cog}(Q)$. So, G is well-defined.

The sequence $0 \to \mathrm{Ker}(f) \xrightarrow{i} V^n \xrightarrow{f} X \to 0$ is exact, hence the sequence $0 \to \mathrm{G}(X) \xrightarrow{\mathrm{G}(f)} \mathrm{G}(V^n) \xrightarrow{\mathrm{G}(i)} \mathrm{Im}(\mathrm{G}(i)) \to 0$ is exact. Since $\mathrm{Im}(\mathrm{G}(i))$ is a subobject of $\mathrm{G}(\mathrm{Ker}(f))$ it follows, by Lemma 3.1, that $\mathrm{Im}(\mathrm{G}(i)) \in \mathrm{Faith}_\delta$.

Because F is w-π_f-*exact* the sequence $0 \to \mathrm{F}(\mathrm{Im}(\mathrm{G}(i))) \stackrel{\mathrm{FG}(i)}{\to} \mathrm{FG}(V^n) \stackrel{\mathrm{FG}(f)}{\to} \mathrm{FG}(X) \to 0$ is exact, hence $\mathrm{FG}(f)$ is an epimorphism.

The diagram

$$\begin{array}{ccc} V^n & \xrightarrow{\zeta_{V^n}} & \mathrm{FG}(V^n) \\ f\downarrow & & \downarrow \mathrm{FG}(f) \\ X & \xrightarrow{\zeta_X} & \mathrm{FG}(X). \end{array}$$

is commutative, so $\zeta_X \circ f = \mathrm{FG}(f) \circ \zeta_{V^n}$. Since ζ_X is an epimorphism, because ζ_{V^n} and $\mathrm{FG}(f)$ are epimorphisms, we have that $X \in \mathrm{Refl}_\zeta$, hence $\mathrm{gen}(V) \cap \mathrm{Faith}_\zeta \subseteq \mathrm{Refl}_\zeta$.

By what we just proved, $\mathrm{F} : \mathrm{cog}(Q) \rightleftarrows \mathrm{gen}(V) \cap \mathrm{Faith}_\zeta : \mathrm{G}$ is a duality. □

Finitistic self-cotilting modules. The following kind of dualities was characterized in [2] and [3].

Theorem 3.6. *The following are equivalent for a pair* $\mathfrak{D}$*:*

a) $\mathrm{F} : \mathrm{cog}(Q) \rightleftarrows \mathrm{pres}(V) \cap \mathrm{Faith}_\zeta : \mathrm{G}$ *is a duality;*

b) i) $\mathrm{cog}(Q) = \mathrm{cop}(Q)$*;*

ii) F *is exact with respect exact sequences* $0 \to X \to Q^n \to Y \to 0$ *with* $Y \in \mathrm{cog}(Q)$. □

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